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GENERALIZED BEAM THEORY APPLIED TO SHEAR STIFFNESS

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Abstract—There is no consensus of opinion on the correct expressions for shear stiffness, even in the apparently simple case of rectangular sections made from homogeneous isotropic material. A general beam theory has been proposed which is applicable to all regular prismatic systems. This has been used to find the appropriate beam-like flexibilities for trusses. The same approach can be used for normal beams, giving values for the shear stiffnesses of various cross-sections as particular results of a general theory embracing torsion, bending, extension and shear of regular prismatic systems.

1. INTRODUCTION

It is perhaps surprising that even the macroscopic behaviour of a beam with a rectangular section made from homogeneous, isotropic, linearly-elastic material is not generally understood. At least four different values for the shear stiffness of such a beam have been suggested, in addition to GA where G is the shear modulus of the material and A is the area of the section. These are

- $GA/1.2$ (Washizu, 1968),
- $GA/1.5$ (Timoshenko and Goodier, 1970),
- $GA(1-v^2)$ (Lowe, 1971),
- $20GA(1+v)/3(8+5v)$ (Donnell, 1978),

where v is Poisson's ratio. Problems arise in defining what the shear behaviour is, and in trying to differentiate the shear deflection from some local rigid-body rotation of the beam.

Consider the behaviour of a thin rectangular beam of depth h in a state of plane stress induced by an exponentially decaying distributed load q, applied to the top surface of the beam, where

$$q = q_0 e^{-az/h}, \tag{1}$$

z being measured along the axis of the beam. A solution to this problem is given by

$$\sigma_{zz} = [A(\sin kx + kx\cos kx) + B(2\cos kx - kx\sin kx)]e^{-kz} \tag{2}$$

$$\tau_{zx} = [Akx\sin kx + B(\sin kx + kx\cos kx)]e^{-kz} \tag{3}$$

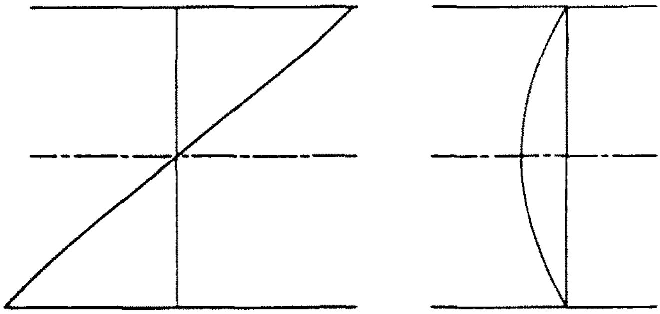
$$\sigma_{xx} = [A(\sin kx - kx\cos kx) + Bkx\sin kx]e^{-kz} \tag{4}$$

where

$$k = a/h \tag{5}$$

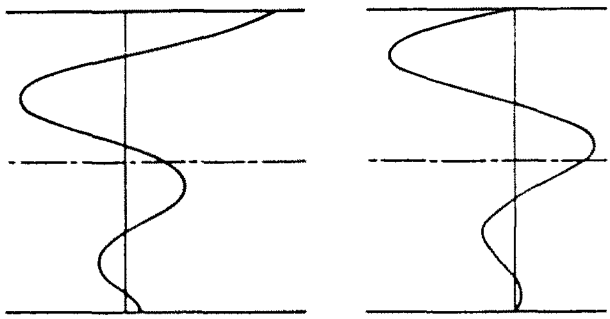
$$A = q_0(\sin a + a\cos a)/(a^2 - \sin^2 a) \tag{6}$$

$$B = -q_0 a\sin a/(a^2 - \sin^2 a) \tag{7}$$



(a) (b)

Fig. 1. Stress distributions for $a = 1$. (a) Axial stress, σ_{zz}. (b) Shear stress, τ_{zx}.



(a) (b)

Fig. 2. Stress distributions for $a = 10$. (a) Axial stress, σ_{zz}. (b) Shear stress, τ_{zx}.

and x is the distance upwards from the base of the beam. Figures 1a and 1b show the bending and shear stress distributions respectively for the case when a is unity. These are close to the linear bending stress and parabolic shear stress distributions normally assumed in the engineering theory of beams. However, suppose that the value of a is increased to 10. The bending and shear stress distributions are now given by Figs 2a and 2b, respectively. Clearly, the ordinary engineering theory of bending is no longer applicable. It will be argued that, in the latter case, the applied loading is varying so rapidly that the beam is unable to settle down to its "characteristic" response. Such a characteristic response may be inferred from St Venant's principle.

A generalization of beam theory was previously proposed (Renton, 1984) which was applicable to any regular prismatic system, including both beams and trusses. The usual starting point for isotropic beam theory, that plane sections remain plane, was not only meaningless in such a general context, but found to be incorrect for certain anisotropic beams (Renton, 1987). Implicit in the use of a beam theory is the assumption that the beam responds in a characteristic way to a resultant load, regardless of the details of the way in which that load is applied. From St Venant's principle, the system should settle down to a characteristic response to a resultant load at large distances from the zone where the load is applied. If the resultant load is a bending moment, torque or axial force, its effect remains constant along the system and so the characteristic response must be one of constant stress and strain. For isotropic beams, it can be shown that this leads to the condition of plane sections remaining plane in bending and under axial force, and to St Venant's theory of torsion.

If a shear force is applied to the end of a beam, the state at large distances from the end cannot be a steady state of stress and strain, as the shear force will induce a linearly-varying moment. Instead, the state is characterized by a bending stress distribution which varies linearly with the moment, plus a constant stress distribution related to the shear force.

Suppose that such a state has been found, giving the necessary resultant moment and shear force and satisfying the equations of equilibrium, compatibility and zero traction on the lateral surfaces of the beam. This must then be the unique characteristic response of the beam at large distances from the end where the shear force is applied. If this were not so, then some other non-decaying state would exist, corresponding to the same resultant end loading. Taking the difference of the two states would then give a non-decaying state of stress corresponding to zero resultant applied end loading, thus contravening St Venant's principle.

The problem is then reduced to finding the appropriate rate of shear deflection from this characteristic response. This must be distinguished from any rigid-body rotation of the bar about some lateral axis, because the shear deflection sought is due solely to an elastic response to a shear force. As will be seen, this may be done by separating the strain energy per unit length into bending and shear components. The latter may then be associated with the work done by the shear force during the shear deflection of a unit length of the beam. The shear force being known, the rate of shear deflection, and hence the shear stiffness, are then determined.

2. SHEAR BEHAVIOUR OF ISOTROPIC BEAMS

The required characteristic behaviour of isotropic beams can be found in Chapter 11 of Timoshenko and Goodier (1970) for example. The stress distribution induced by the shear force S at a distance z along the beam shown in Fig. 3 is given by

$$\sigma_{zz} = \frac{Szx}{I} \tag{8}$$

$$\tau_{xz} = \frac{\partial \phi}{\partial y} - \frac{Sx^2}{2I} + f(y) \tag{9}$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} \tag{10}$$

where ϕ is a stress function which satisfies the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1+\nu}\frac{Sy}{I} - \frac{\mathrm{d}f}{\mathrm{d}y} \tag{11}$$

on the cross-section, $f(y)$ is a function such that

$$\frac{\partial \phi}{\partial s} = \left[\frac{Sx^2}{2I} - f(y)\right]\frac{\mathrm{d}y}{\mathrm{d}s} = 0 \tag{12}$$

on the boundary of the cross-section, where s is measured around the boundary, x and y

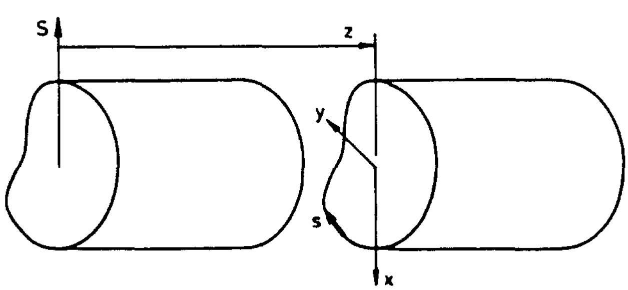


Fig. 3. Shear force acting on a beam.

are the principal axes of the cross-section through its centroid and I is the second moment of area of the cross-section about the y axis. Without loss of generality, ϕ can be taken as identically zero on the boundary.

The bending strain energy per unit length, U_B, is given by the energy associated with the linearly-varying components of stress and strain, or from (8),

$$U_B = \frac{1}{2}\int_A \sigma_{zz}\varepsilon_{zz}\,dA = \frac{(Sz)^2}{2EI} \tag{13}$$

where A is the area of the cross-section and E is Young's modulus. The work done per unit length by the bending moment, M say, is

$$W_B = \tfrac{1}{2}M\psi \tag{14}$$

where ψ is the rate of change of flexural rotation. Defining a bending stiffness K_B and a bending flexibility F_B such that

$$M = K_B\psi, \quad \psi = F_B M \tag{15}$$

then (14) becomes

$$W_B = \tfrac{1}{2}M^2F_B. \tag{16}$$

Now at a distance z along the beam

$$M = Sz \tag{17}$$

so that on comparing the bending strain energy per unit length with the bending work done per unit length, from (13) to (17),

$$F_B = \frac{1}{EI}$$

giving the usual value of EI for the bending stiffness. Exactly the same result is given by examining the constant characteristic response to an end moment M, when only the stress

$$\sigma_{zz} = \frac{Mx}{I}$$

is induced. Note that this solution is given by the differential of the shear solution with respect to z. The same relationship holds for all other cases including anisotropic beam problems; see for example Lekhnitskii (1981).

The remaining strain energy per unit length, U_S, is associated with shear. From (9) and (10), this is given by

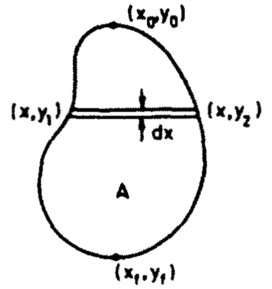


Fig. 4. Integration over the cross-section.

$$U_S = \frac{1}{2G}\int_A (\tau_{xz}^2 + \tau_{yz}^2)\,\mathrm{d}A = \frac{1}{2G}\int_A \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] + \left[-\frac{Sx^2}{2I} + f\right]\left[2\frac{\partial\phi}{\partial y} - \frac{Sx^2}{2I} + f\right]\mathrm{d}A. \quad (18)$$

Now from Fig. 4,

$$\int_A \left(\frac{\partial\phi}{\partial y}\right)^2 \mathrm{d}A = \int_{x_0}^{x_f}\int_{y_1}^{y_2}\left(\frac{\partial\phi}{\partial y}\right)\left(\frac{\partial\phi}{\partial y}\right)\mathrm{d}y\,\mathrm{d}x = \int_{x_0}^{x_f}\left(\left|\phi\frac{\partial\phi}{\partial y}\right|_{y_1}^{y_2} - \int_{y_1}^{y_2}\phi\frac{\partial^2\phi}{\partial y^2}\mathrm{d}y\right)\mathrm{d}x = -\int_A \phi\frac{\partial^2\phi}{\partial y^2}\mathrm{d}A \quad (19)$$

because ϕ is zero on the boundary. Likewise,

$$\int_A \left(\frac{\partial\phi}{\partial x}\right)^2 \mathrm{d}A = -\int_A \phi\frac{\partial^2\phi}{\partial x^2}\mathrm{d}A \quad (20)$$

so that from (19) and (20),

$$\int_A \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]\mathrm{d}A = -\int_A \phi\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\mathrm{d}A = \int_A \phi\left(\frac{\mathrm{d}f}{\mathrm{d}y} - \frac{v}{1+v}\frac{Sy}{I}\right)\mathrm{d}A \quad (21)$$

from (11). Also

$$\int_A 2\frac{\partial\phi}{\partial y}\left[f - \frac{Sx^2}{2I}\right]\mathrm{d}A = \int_{x_0}^{x_f}\left[\left|2\phi\left(f - \frac{Sx^2}{2I}\right)\right|_{y_1}^{y_2} - \int_{y_1}^{y_2}2\phi\frac{\mathrm{d}f}{\mathrm{d}y}\mathrm{d}y\right]\mathrm{d}x = -\int_A 2\phi\frac{\mathrm{d}f}{\mathrm{d}y}\mathrm{d}A \quad (22)$$

again, because ϕ is zero on the boundary. Then from (18) to (22),

$$U_S = \frac{1}{2G}\int_A \left[f - \frac{Sx^2}{2I}\right]^2 - \phi\left(\frac{\mathrm{d}f}{\mathrm{d}y} + \frac{v}{1+v}\frac{Sy}{I}\right)\mathrm{d}A. \quad (23)$$

Now f and ϕ will be proportional to S, so that it is convenient to define

$$F = \frac{If}{S}, \quad \Phi = \frac{I\phi}{S} \tag{24}$$

where F and Φ are not functions of S. Then if K_S and F_S are the shear stiffness and shear flexibility of the section under the action of S, the work done per unit length by the shear force on the section due to the shear deformation of the section, is

$$W_S = \tfrac{1}{2}S^2 F_S \tag{25}$$

[cf. (16)]. Then on comparing the work done by the shear force per unit length with the shear strain energy per unit length, from (23) to (25),

$$\frac{1}{K_S} = F_S = \frac{1}{GI^2}\int_A \left(F - \frac{1}{2}x^2\right)^2 - \Phi\left(\frac{dF}{dy} + \frac{vy}{1+v}\right) dA. \tag{26}$$

3. SHEAR STIFFNESSES OF VARIOUS SECTIONS

The following results are derived from the analyses in Chapter 11 of Timoshenko and Goodier (1970).

3.1. *Rectangular sections*

For a rectangular section of depth $2a$ and breadth $2b$,

$$F = \tfrac{1}{2}a^2, \quad \Phi = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{2m+1,n} \cos\left(\frac{(2m+1)\pi x}{2a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

where

$$a_{2m+1,n} = \left(\frac{v}{1+v}\right) \frac{8b(-1)^{m+n}}{\pi^4(2m+1)n[(2m+1)^2/(2a)^2 + (n/b)^2]}$$

so that from (26),

$$F_S = \frac{1}{GI^2}\int_{-b}^{b}\int_{-a}^{a} \frac{1}{4}(a^2 - x^2)^2 - \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\frac{vy}{1+v}\right) a_{2m+1,n}\cos\left(\frac{(2m+1)\pi x}{2a}\right)\sin\left(\frac{n\pi y}{b}\right) dx\,dy$$

where

$$I = \tfrac{4}{3}ba^3, \quad A = 4ba.$$

This gives

$$\frac{1}{K_S} = F_S = \frac{1}{GA}\left[\frac{6}{5} + \left(\frac{v}{1+v}\right)^2 \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{144(b/a)^4}{\pi^6(2m+1)^2 n^2[(2m+1)^2(b/2a)^2 + n^2]}\right].$$

When v is zero, or as b/a tends to zero, this gives

$$\frac{1}{K_S} = F_S = \frac{1.2}{GA} \tag{27}$$

which corresponds to the result given by Washizu (1968) and Young (1989). Using the results

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}=\frac{\pi^2}{8},\quad \sum_{m=0}^{\infty}\frac{1}{(2m+1)^4}=\frac{\pi^4}{96},\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\quad \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90},$$

as b/a tends to infinity, the values tend towards

$$\frac{1}{K_S}=F_S=\frac{1}{GA}\left[1.2+\left(\frac{v}{1+v}\right)^2\left(\frac{b}{a}\right)^2\right]. \tag{28}$$

Intermediate values are given by

$$\frac{1}{K_S}=F_S=\frac{1}{GA}\left[1.2+C_1\left(\frac{v}{1+v}\right)^2\left(\frac{b}{a}\right)^4\right]\qquad \frac{b}{a}\leqslant 1.0 \tag{29}$$

$$\frac{1}{K_S}=F_S=\frac{1}{GA}\left[1.2+C_2\left(\frac{v}{1+v}\right)^2\left(\frac{b}{a}\right)^2\right]\qquad \frac{b}{a}\geqslant 1.0 \tag{30}$$

where

$$C_1=0.2\quad (b/a\ll 1.0),\quad C_2=1.0\quad (b/a\gg 1.0)$$

and some other values are given by

b/a	C_1	C_2
0.1	0.1939	—
0.2	0.1878	—
0.5	0.1695	—
1.0	0.1392	0.1392
2.0	—	0.3511
5.0	—	0.6699
10.0	—	0.8229

3.2. *Circular sections*

For a circular section of radius R,

$$F=\tfrac{1}{2}(R^2-y^2),\quad \Phi=-\frac{1-2v}{8(1+v)}(R^2-r^2)y$$

where in terms of polar coordinates (r,θ),

$$y=r\cos\theta.$$

Then

$$\frac{1}{K_S}=F_S=\frac{1}{GI^2}\int_0^R\int_0^{2\pi}\frac{1}{2}(R^2-r^2)^2-\frac{1+2v}{8(1+v)}(R^2-r^2)r^2\cos^2\theta\left(1-\frac{v}{1+v}\right)r\,d\theta\,dr$$

$$=\frac{1}{6GA}\left[7+\left(\frac{v}{1+v}\right)^2\right] \tag{31}$$

where

$$I = \frac{\pi R^4}{4}, \quad A = \pi R^2.$$

This may be compared with the value of $10/(9GA)$ quoted by Young (1989).

3.3. *Elliptic sections*

For an elliptic section with semi-axes of lengths a and b in the directions of the x and y axes respectively,

$$\Phi = \frac{(1+\nu)a^2+\nu b^2}{2b^2(1+\nu)(3a^2+b^2)}(x^2b^2+y^2a^2-a^2b^2)y$$

$$F = \frac{a^2(b^2-y^2)}{2b^2}$$

so that

$$\frac{1}{K_S} = F_S = \frac{1}{GI^2}\int_A \frac{a^4}{4}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2 + \frac{(1+\nu)^2a^4-\nu^2b^4}{2(1+\nu)^2(3a^2+b^2)}\frac{a^2}{b^2}y^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\mathrm{d}A.$$

This integration can be carried out in two ways. One is to use the coordinate transformation

$$x = p\sinh\beta\cos\alpha, \quad y = p\cosh\beta\sin\alpha$$

where

$$p^2 = b^2-a^2, \quad \mathrm{d}A = p^2(\cos^2\alpha+\sinh^2\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta$$

and the limits of integration are

$$\alpha = 0 \quad \text{to} \quad 2\pi, \quad \beta = 0 \quad \text{to} \quad \sinh^{-1}(a/p).$$

Alternatively, a double coordinate transformation can be used, where

$$X = x, \quad Y = \frac{ay}{b}, \quad \mathrm{d}A = \frac{b}{a}\mathrm{d}X\,\mathrm{d}Y$$

which transforms the boundary into a circle of radius a, and then polar coordinates (r, θ) can be used, where

$$X = r\cos\theta, \quad Y = r\sin\theta, \quad \mathrm{d}X\,\mathrm{d}Y = r\,\mathrm{d}\theta\,\mathrm{d}r.$$

This yields the same result more readily, giving

$$\frac{1}{K_S} = F_S = \frac{1}{6GA}\left[6+\frac{2(a^2+b^2)}{3a^2+b^2}+\left(\frac{\nu}{1+\nu}\right)^2\frac{4b^4}{a^2(3a^2+b^2)}\right]. \tag{32}$$

It will be seen that in the particular case when $a = b$, this reduces to the expression for a circular section given by (31).

3.4. *A triangular section*

A closed form of solution exists for an equilateral triangular section of side $2\sqrt{3}a$ made from incompressible material ($\nu = 0.5$) with a shear force acting parallel to one side. Then

$$F = \tfrac{1}{6}(2a+y)^2, \quad \Phi = \tfrac{1}{6}[x^2 - \tfrac{1}{3}(2a+y)^2](y-a)$$

where the side parallel to the shear force is given by $y = a$. After some simplification, this gives

$$\frac{1}{K_S} = F_S = \frac{1}{36GI^2}\int_A [x^2 - \tfrac{1}{3}(2a+y)^2](9x^2 - 7y^2 - 8a^2 - 12ay)\,\mathrm{d}A = \frac{4}{3GA} \tag{33}$$

where

$$I = \frac{3\sqrt{3}a^4}{2}, \quad A = 3\sqrt{3}a^2.$$

This may be compared with the value of $6/(5GA)$ quoted by Young (1989).

4. SLOPE-DEFLECTION EQUATIONS MODIFIED FOR SHEAR

In conformity with the methods used to find the bending and shear stiffnesses of beams, the slope-deflection equations may also be determined from the strain energy of a beam. Consider a cantilever of length l, fixed at its right-hand end, B, and loaded by a clockwise moment M_A and upwards shear force F_A at its left-hand end, A. The strain energy stored in the beam is then

$$U_l = \frac{1}{2}\int_0^l \frac{1}{EI}(M_A + F_A z)^2\,\mathrm{d}z + \frac{1}{2}\int_0^l \frac{1}{K_S} F_A^2\,\mathrm{d}z. \tag{34}$$

The clockwise rotation, θ_A^0, and the upwards displacement, δ_A^0, of end A are given by

$$\theta_A^0 = \frac{\partial U}{\partial M_A} = \frac{M_A l}{EI} + \frac{F_A l^2}{2EI} \tag{35}$$

$$\delta_A^0 = \frac{\partial U}{\partial F_A} = \frac{M_A l^2}{2EI} + \frac{F_A l^3}{3EI}(1+s/4) \tag{36}$$

where

$$s = \frac{12EI}{K_S l^2}. \tag{37}$$

These equations can be inverted to give M_A and F_A in terms of θ_A^0 and δ_A^0,

$$(1+s)M_A = \frac{EI}{l}\left[(4+s)\theta_A^0 - \frac{6}{l}\delta_A^0\right] \tag{38}$$

$$(1+s)F_A = \frac{EI}{l}\left[-\frac{6}{l}\theta_A^0 + \frac{12}{l^2}\delta_A^0\right]. \tag{39}$$

Suppose that a rigid-body movement is given to the bar and the fixed end as a whole, so that the right-hand end now has a clockwise rotation θ_B and an upwards displacement δ_B. The rotation, θ_A, and the displacement, δ_A, of the left-hand end are now

$$\theta_A = \theta_A^0 + \theta_B, \quad \delta_A = \delta_A^0 + \theta_B l + \delta_B. \tag{40}$$

Using these relationships to substitute for θ_A^0 and δ_A^0 in (38) and (39) gives the general results

$$M_A = \frac{EI}{l(1+s)}\left[(4+s)\theta_A + (2-s)\theta_B - \frac{6}{l}(\delta_A - \delta_B)\right] \tag{41}$$

$$F_A = \frac{EI}{l(1+s)}\left[-\frac{6}{l}(\theta_A + \theta_B) + \frac{12}{l^2}(\delta_A - \delta_B)\right]. \tag{42}$$

The clockwise end moment M_B and upwards shear force F_B at the end B can be determined from statics, giving

$$M_B = \frac{EI}{l(1+s)}\left[(2-s)\theta_A + (4+s)\theta_B - \frac{6}{l}(\delta_A - \delta_B)\right] \tag{43}$$

$$F_B = \frac{EI}{l(1+s)}\left[\frac{6}{l}(\theta_A + \theta_B) - \frac{12}{l^2}(\delta_A - \delta_B)\right]. \tag{44}$$

These equations reduce to their more normal forms on taking s as zero, which corresponds to an infinite shear stiffness.

If the end moments and shear forces are applied by axial and shear stress distributions of the characteristic form, then the slope-deflection equations are exact solutions in terms of the small deflection theory of elasticity. If the end loadings are not of the characteristic form, then from St Venant's principle, they will decay towards the characteristic forms away from the ends, and the overall behaviour of the beam, as given by the slope-deflection equations, will remain an approximation.

5. THE GENERAL THEORY

It should be borne in mind that the above results were obtained for homogeneous isotropic beams, where the flexural distortion is directly related to the bending moment induced by the shear force and the shear distortion directly related to the shear force. The lack of coupling means that the bending and shear stiffnesses are simply the reciprocals of the corresponding flexibilities. More generally, a torque may produce flexure in an anisotropic beam (Renton, 1987), or an axial force produce shear distortion in a truss (Renton, 1984). In the latter paper, shear stiffnesses for trusses were found by similar energy methods to those used above.

Allowing for all such possible couplings, the flexibility equations relating resultant loads to their corresponding distortions become

$$\begin{bmatrix} \theta \\ \psi_x \\ \psi_y \\ \varepsilon \\ \gamma_x \\ \gamma_y \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} & f_{46} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} & f_{56} \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} \end{bmatrix} \begin{bmatrix} T \\ M_x \\ M_y \\ P \\ S_x \\ S_y \end{bmatrix} \tag{45}$$

where $\{\theta, \psi_x, \psi_y, \varepsilon, \gamma_x, \gamma_y\}$ are the rates of twist, flexure about the x and y axes, axial extension and shear distortion in the x and y directions and $\{T, M_x, M_y, P, S_x, S_y\}$ are the local torque, moments about the x and y axes, axial force and shear forces in the x and y directions. To find the corresponding stiffnesses, it is then necessary to invert the whole flexibility matrix.

In the general case, it becomes necessary to define what is meant by the rates of twist, flexure, axial extension and shear, as no single cross-section necessarily typifies the overall

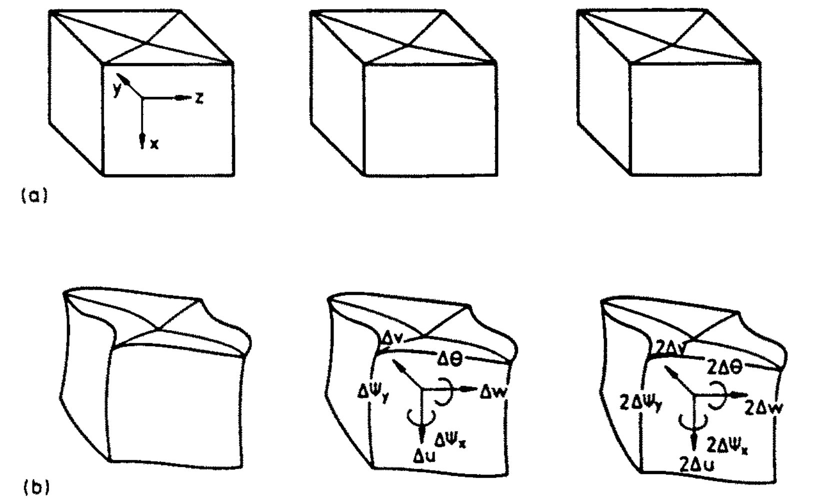


Fig. 5. Three adjacent segments of a prismatic system. (a) Before loading. (b) After loading.

response. Figure 5 shows three adjacent identical segments of the system. These are not necessarily brick-shaped as shown, but could be similar lengths of a castellated beam or bays of a truss. After loading with a bending moment, torque or axial force, the characteristic response is such that the corresponding strains in each segment are the same. The deflections in adjacent segments then differ only by some relative rigid-body displacement of the two. Moreover, this relative motion will be the same between all adjacent segments, so that for example the motion of the third segment relative to the first is twice that of the second relative to the first. This means that constant rates of torsion, flexure, extension and shear are expressed by such relative rigid-body rotations and displacements, even when the system is regular rather than uniform. Thus (45) relates the complete set of such relative motions to the complete set of resultant loads.

It follows from Betti's reciprocal theorem that the matrix is symmetrical ($f_{ij} = f_{ji}$). Suppose that in (45) the distortion and load vectors are written as $\{d_1, d_2, d_3, d_4, d_5, d_6\}$ and $\{P_1, P_2, P_3, P_4, P_5, P_6\}$, respectively. The equation then takes the form

$$d_i = \sum_{j=1}^{6} f_{ij} P_j \quad (i = 1-6). \tag{46}$$

The work done per unit length, W, by the loads in deforming the system is then

$$W = \frac{1}{2}\sum_{i=1}^{6} P_i d_i = \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6} P_i f_{ij} P_j. \tag{47}$$

This is equal to the strain energy per unit length, U. Then

$$\frac{\partial^2 U}{\partial P_i \partial P_j} = \frac{\partial^2 W}{\partial P_i \partial P_j} = \frac{1}{2}(f_{ij}+f_{ji}) = f_{ij}. \tag{48}$$

Then if the strain energy per unit length can be expressed as a quadratic in terms of the loads P_i, the flexibility coefficients can be found.

Even in the most general case, a limited degree of uncoupling can be achieved. A suitable orientation of the x and y axes can be chosen so that a moment about one axis does not produce flexure about the other ($f_{23} = f_{32} = 0$). The lines of action of the forces could be redefined so that the axial force P acts along a line parallel to the z axis, through points with coordinates (x_0, y_0), and the shear forces taken to act through points with coordinates (x_c, y_c). This means that the torque and bending moments acting have to be redefined as T^c, M_x^0 and M_y^0, where, to give the same resultant loading, they are related to the original loading by

$$T = T^c - y_c S_x + x_c S_y \tag{49}$$

$$M_x = M_x^0 + y_0 P \tag{50}$$

$$M_y = M_y^0 - x_0 P. \tag{51}$$

Consider each segment of the system to have a reference plane initially perpendicular to the z axis which undergoes the same overall rotations and displacements as the segment. A new axial strain ε^0 and shear strains γ_x^c and γ_y^c can be defined in terms of the relative axial displacements of the points (x_0, y_0) and shear displacements of the points (x_c, y_c) on these planes. From geometrical considerations, these are

$$\varepsilon^0 = y_0 \psi_x - x_0 \psi_y + \varepsilon \tag{52}$$

$$\gamma_x^c = -y_c \theta + \gamma_x \tag{53}$$

$$\gamma_y^c = x_c \theta + \gamma_y. \tag{54}$$

Equations (49)–(54) can be used in conjunction with (45) to derive a new matrix of flexibility coefficients f_{ij}^* relating the load vector $\{T^c, M_x^0, M_y^0, P, S_x, S_y\}$ to the distortion vector $\{\theta, \psi_x, \psi_y, \varepsilon^0, \gamma_x^c, \gamma_y^c\}$. The symmetry of the flexibility matrix is retained. In particular, by choosing

$$x_c = -\frac{f_{16}}{f_{11}}, \quad y_c = \frac{f_{15}}{f_{11}} \tag{55}$$

$$x_0 = \frac{f_{14} f_{22} - f_{24} f_{12}}{f_{22} f_{33} - f_{23} f_{32}} \tag{56}$$

$$y_0 = \frac{f_{14} f_{23} - f_{24} f_{33}}{f_{22} f_{33} - f_{23} f_{32}} \tag{57}$$

f_{15}^*, f_{16}^*, f_{24}^* and f_{34}^* (and therefore f_{51}^*, f_{61}^*, f_{42}^* and f_{43}^*) are zero. These uncoupling conditions can be thought of as defining a "centroid" with coordinates (x_0, y_0) and a "shear centre" with coordinates (x_c, y_c) in the most general case.

Further uncoupling can be inferred from any summetry of the system. For example, if the system is invariant under inversion of the z axis, all the remaining coupling terms, with the possible exception of f_{56} and f_{65}, will be zero. The cases examined in Section 3 exhibit no coupling, and the shear deformation is related to the deflection of the original z axis of the section.

6. CONCLUDING REMARKS

Shear stiffnesses for a variety of isotropic bars with simple cross-sections have been found, using a generalization of engineering beam theory. All these results give shear stiffnesses which are less than those quoted by Young (1989). Unlike any of the previous results listed, the expression for the shear flexibility of a rectangular section with a constant cross-sectional area shows that it increases in proportion to $(b/a)^2$ as the section becomes a very thin, flat strip. However, since the bending flexibility of the section will increase in proportion to b/a, the result is not so surprising. Also, from (32), it will be seen that a very broad, shallow elliptic section exhibits similar shear behaviour.

In only the case of a triangular section was the solution found in terms of a specific value of Poisson's ratio. In all other cases, the shear stiffness can be written in the form

$$K_S = \frac{GA}{B + C\left(\frac{\nu}{1+\nu}\right)^2}$$

where B and C are positive constants depending on the geometry of the section, and B is greater than unity. It may be conjectured that this is the general form of the result.

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